

Convergence to equilibrium for a thin film equation on a cylindrical surface

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Abstract

The degenerate parabolic equation $u_t + \partial_x[u^3(u_{xxx} + u_x - \sin x)] = 0$ models the evolution of a thin liquid film on a stationary horizontal cylinder. It is shown here that for each mass there is a unique steady state, given by a droplet hanging from the bottom of the cylinder that meets the dry region with zero contact angle. The droplet minimizes the associated energy functional and attracts all strong solutions that satisfy certain energy and entropy inequalities, including all positive solutions. The distance of solutions from the steady state cannot decay faster than a power law.

Keywords Thin liquid film; coating flow; strong solutions; steady state; symmetrization; energy; entropy method; Lyapunov stability; power-law decay.

AMS Subject Classification: 35K25; 35K35; 35Q35; 37L05; 76A20.

1 Introduction and description of the results

Degenerate fourth order parabolic equations of the form

$$u_t + \nabla \cdot (u^n \nabla \Delta u) + \text{lower order terms} = 0 \quad (1.1)$$

are used to model the evolution of thin liquid films on solid surfaces. Here, $u(x, t)$ describes the thickness of the fluid at time t at the point x , the fourth derivative term models the surface tension, and the exponent $n > 0$ is determined by the boundary condition between the liquid and the solid. The equations were derived from the underlying free-boundary-value problem for the Navier-Stokes equation by the lubrication approximation, which is valid if the film is relatively thin.

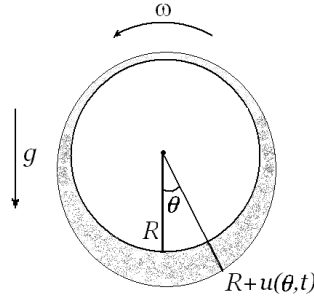


Figure 1: Thin liquid film on the outer surface of a horizontal cylinder.

An interesting example is Pukhnachev's model for a thin liquid film on a rotating horizontal cylinder [33], as shown in Figure 1. The model equation is

$$u_t + \partial_\theta \left[\frac{1}{3} \frac{\sigma}{\rho \nu R^4} u^3 (u_{\theta\theta\theta} + u_\theta) - \frac{1}{3} \frac{g}{\nu R} u^3 \sin \theta + \omega u \right] = 0. \quad (1.2)$$

Here, $\theta \in \mathbb{R}/(2\pi\mathbb{Z})$ denotes the angle measured from the bottom of the cylinder, the film is assumed to be uniform in the axial direction, and inertia is ignored. The power $n = 3$ indicates a *no-slip* boundary condition between the liquid and the solid. The other physical parameters of the system are the surface tension (σ), the viscosity (ν), the acceleration of gravity (g), the density of the fluid (ρ), the radius of the cylinder (R), and the rotation speed (ω). Note that the surface tension appears with an additional lower-order term that corrects for the curvature of the cylinder. Eq. (1.2) refines an earlier model of Moffatt [30] that neglects surface tension. Numerical and asymptotical analysis of Pukhnachev's equation along with numerous open questions can be found in [2, 3, 21], and linearizations about steady states are examined analytically and numerically in [13, 19].

We will show that the long-time behaviour of Pukhnachev's model on a *non-rotating* cylinder is controlled by steady states. Our results imply that

- Eq. (1.2) with $\omega = 0$ has for every given mass a unique nonnegative steady state;
- the steady state minimizes energy and attracts all solutions of finite entropy;
- the distance of solutions from the steady state decays no faster than a power law $\sim t^{-\frac{2}{3}}$.

Note that solutions of finite entropy are almost everywhere strictly positive. The steady state has the shape of a shallow drop hanging from the bottom of the cylinder, with a dry region at the top that it meets at zero contact angle, see Figure 2 below. We suspect that the distance from the steady state actually behaves as $t^{-\frac{1}{3}}$. This conjecture is supported by simulations, and by analogy with aggregation processes such as late-stage grain growth in alloys [27] and the formation of drops at a faucet, where the growth of the grain or droplet is limited by the rate of mass transfer through a region of low density to the region of accumulation.

Our results are motivated by the work of Carrillo and Toscani, who proved global convergence to self-similar solutions for the thin film equation

$$u_t + \partial_x [u^n u_{xxx}] = 0 \quad (1.3)$$

on the real line with $n = 1$ [10]. In contrast with the recent precise convergence results of Giacomelli, Knüpfer and Otto [16], our conclusions are qualitative and global. We work in a large class of non-negative strong solutions that contains all positive classical solutions, and we do not assume (and do not prove) uniqueness of the solutions. Our lower bounds on the distance from the steady state should be compared with results of Carlen and Ulusoy [11], who showed for Eq. (1.3) with $n = 1$ that the distance from the self-similar solution satisfies a power-law *upper bound*. We will give a more detailed description at the end of this section.

Thin liquid films have been the subject of rigorous mathematical analysis since the pioneering article of Bernis and Friedman [5]. A vast body of papers is dedicated to existence of solutions, regularity, long-time behaviour, finite-time blow-up, and the interface between wet ($u > 0$) and dry ($u = 0$) regions, see for example [1, 4, 7, 8, 14, 31] and references therein. An even larger part of the literature studies the properties of physically relevant solutions through asymptotic expansions, numerical analysis, and laboratory experiments.

A fundamental question is *well-posedness*: which initial values give rise to unique nonnegative solutions that depend continuously on the data? The difficulty is that solutions of fourth order parabolic equations generally do not satisfy a maximum principle, and linearization leads to semigroups that do not preserve positivity. To give a simple example, the function $u(x, t) = 1 + t \cos(x)$, which develops negative values after $t = 1$, solves Eq. (1.4) (given below) with $n = 0$ and $\alpha = 1$. Bernis and Friedman proved that initial-value problems in one space dimension have weak solutions in suitable function spaces [5]. A far-reaching technical contribution was their use of *energy* and *entropy* functionals that decrease along solutions. Still, after twenty years, uniqueness remains an open problem.

There are many questions surrounding *steady states*: Are they uniquely determined by their mass, and if not, how many are there? Are they strictly positive, and if not, what is the contact angle between wet and dry regions [25]? Under what conditions are they stable, do they attract all bounded solutions, and what is the rate of convergence? Since energy decreases along solutions, we expect that steady states should correspond to critical points of the energy, that solutions should converge to steady states, and that minimizers of the energy should be asymptotically stable. However, in the absence of a proper well-posedness theory, the proof of these statements requires more than a standard application of Lyapunov's principle (as stated, for example, in [18]).

One strategy for proving convergence to equilibrium is to use entropy in place of energy. The basic thin film equation (1.3) has a family of entropy functionals of the form $S_\beta(u) = \int_\Omega u^{-\beta} dx$ that decrease along solutions. For $\beta = n - 2$, this was established by Bernis and Friedman, for $\beta = n - \frac{3}{2}$ it is due to Kadanoff (see [6]), and the range $n - 3 < \beta < n - \frac{3}{2}$ was developed independently in [4, 7]. Other families of entropies have since been discovered [20, 24]. In their classical papers, Beretta, Bertsch, Dal Passo and Bertozzi, Pugh used these entropies to show that solutions of Eq. (1.3) with $n > 0$ on an interval become positive after a finite time and converge uniformly to their mean [4, 7]. Remarkably, this convergence holds for a very broad class of weak solutions, about which little else is known. Several works over the last decade have combined energy and entropy methods by deriving coupled inequalities for the energy and entropy dissipation. In this way, Tudorascu proved exponential convergence to the mean for thin films on finite intervals [35], and Carrillo, Toscani [10] and Carlen, Ulusoy [11] have proved global (power-law) convergence to self-similar solutions of the thin equation with $n = 1$ on the real line [10, 11].

A promising approach, first pursued by Otto [31], treats thin-film equations as *gradient flows* that evolve by steepest descent on the space of nonnegative functions of a given mass, endowed with a suitable metric. Since Eq. (1.1) has the form $u_t = \left[u^n \left(\frac{\delta E}{\delta u} \right)_x \right]_x$, a formal computation suggests certain

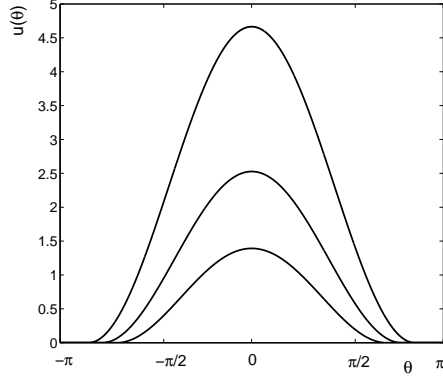


Figure 2: The energy minimizing steady state for Eq. (1.4) with $\alpha = 1$ and initial data $u_0 = 0.5, 1, 2$. The minimizer is concentrated if the mass is small, and spreads out over $(-\pi, \pi)$ as the mass increases.

variants of the Wasserstein metric. These metrics measure the distance between two mass densities as the cost of an optimal transportation plan that pushes one density forward to the other. Gradient flow methods have been very successful for the porous medium and fast diffusion equations, which are second-order degenerate parabolic equations that share many features of thin films. There also has been some progress on thin film equations with $n = 1$, where the metric is just the standard optimal transportation distance [31]. Thin-film equations with $n > 1$ have proved resistant to this approach, because the geometry of the relevant Wasserstein distance with mobility is not well understood, and the thin film energy is not geodesically convex [9].

Yet another strategy is to linearize about the steady state. This linearization is delicate if the steady state has a dry region. Recently, Giacomelli, Knüpfer and Otto have developed a technique specifically for droplets with zero contact angles [16]. They prove well-posedness and convergence to the steady state for initial values in a singularly weighted Sobolev space that forces the solution to vanish to a certain order at the contact point. So far, these techniques have been developed for Eq. (1.3) with $n = 1$.

Outline of the paper. We study the role of energy-minimizing steady states for the dynamics of

$$u_t + \partial_x \left[u^n (u_{xxx} + \alpha^2 u_x - \sin x) \right] = 0, \quad x \in \Omega = \mathbb{R}/(2\pi\mathbb{Z}) \quad (1.4)$$

with $n > 0$ and $\alpha > 0$. Each of these equations describes the evolution of a thin film. As in Eqs. (1.1) and (1.3), the exponent n is related to the boundary condition between the liquid and the solid, the first two terms in the parentheses model surface tension, and the third term accounts for gravitational drainage. The $\alpha^2 \partial_x [u^n \partial_x u]$ term is reminiscent of the porous-medium equation, but appears with the opposite sign, resulting in a long-wave instability [8]. The units of time and length are scaled so that the surface tension and gravitational terms appear with coefficient one. The parameter $\alpha > 0$ is a geometric constant, with $\alpha = 1$ for a horizontal cylinder. Values of $\alpha > 1$ appear when inertial effects are taken into account [23, Eq. (2.3)]. In general, the geometric coefficient is a function that depends on the curvature of the surface [29, Eqs. (64) and (68)]. We are mostly interested in the case where $n = 3$ and $\alpha = 1$, which corresponds to Pukhnachev's model at zero rotation speed, but find it illuminating to consider also other values of n and α .

Many competing definitions of weak solutions have been proposed. Following Bernis and Friedman, we define a *strong solution* of Eq. (1.4) on a finite time interval $(0, T)$ to be a nonnegative function $u \in L^2((0, T), H^2(\Omega))$ that satisfies

$$\int_0^T \int_{\Omega} \left\{ u \phi_t - (u_{xx} + \alpha^2 u + \cos x)(u^n \phi_x)_x \right\} dx dt = 0 \quad (1.5)$$

for every smooth test function ϕ with compact support in $(0, T) \times \Omega$. Such solutions are believed to be unique. We will consider strong solutions that exist for all $t > 0$ and satisfy additional bounds on the energy and entropy (see Section 4). In particular, these solutions are almost everywhere strictly positive.

Our main results concern the convergence of solutions to steady states. We start from the *energy* for Eq. (1.4), given by

$$E(u) = \frac{1}{2} \int_{\Omega} (u_x^2 - \alpha^2 u^2) dx - \int_{\Omega} u \cos x dx, \quad (1.6)$$

where the first integral accounts for the surface tension, and the second integral for the gravitational potential energy. Formally, the energy decreases according to $\frac{dE(u)}{dt} = -D(u)$, where

$$D(u) = \int_{\{u>0\}} u^n (u_{xxx} + \alpha^2 u_x - \sin x)^2 dx \quad (1.7)$$

is the *dissipation* associated with Eq. (1.4).

Note that the energy is not convex for $\alpha > 1$, and not bounded below in H^1 for $\alpha \geq 1$. Nevertheless, we show in Section 2 that for every choice of $\alpha > 0$ the energy has a unique minimizer among *nonnegative* H^1 -functions of a given mass M . We denote this minimizer by u^* . The shape of u^* (depending on α and the mass) is described precisely in Theorem 2.4. The minimization problem is complicated by the fact that the minimizers can have dry regions, where the Euler-Lagrange equation is weakened to an inequality. We combine a careful analysis of the variational inequality with symmetrization techniques.

In Section 3, we show that u^* is the unique steady state of Eq. (1.4) with zero dissipation when $\alpha \leq 1$. For $\alpha > 1$, we find also saddle points, as well as a continuum of two-droplet steady states that are not critical points for the energy in L^2 . Since the definition of strong solutions forces steady states to meet any dry spot at a zero contact angle, Theorem 2.4 implies in particular that such steady states exist if $M(1 - \alpha^2) \leq 2\pi$. For each value of α and M , there is also a continuum of time-independent solutions of Eq. (1.4) with compact support and positive contact angles, which are analogous to the steady states in [25]. Their role in the long-term evolution of positive solutions remains open.

Section 4 contains the main convergence result, Theorem 4.3. We show that the energy minimizer u^* is a dynamically stable, locally attractive steady state of Eq. (1.4). For $\alpha \leq 1$, we show that all solutions that satisfy certain energy and entropy inequalities converge to u^* . For $\alpha > 1$, the convergence holds on a sub-level set of the energy that contains no other strong steady states.

The entropy methods of [4, 7, 10, 11, 35] do not apply to Eq. (1.4), because the entropy can increase as well as decrease along solutions. For steady states with dry regions, the entropy is not even finite. We combine Lyapunov's method with a linear bound on the growth of the entropy to produce a sequence of times along which the solution converges weakly to a steady state, using a recent argument of [12]. We pass to convergence in norm along the full solution by proving a local coercivity estimate on the energy near the minimizer (which becomes global for $\alpha \leq 1$).

Finally, in Section 5, we turn to the rate of convergence. We show that convergence to the steady state is exponential whenever u^* is strictly positive. This extends the results of [4, 7, 35] to examples where the steady state is non-constant (but note that our proof uses energy in place of entropy). When u^* has a dry region and $n > \frac{3}{2}$, we show that solutions cannot approach it faster than a power law $\sim t^{-\frac{2}{2n-3}}$. In the proof, we show that Kadanoff's entropy $S_{n-\frac{3}{2}}$ grows at most linearly.

All our results are easily adapted to the long-wave stable case of Eq. (1.4) where the $\alpha^2 \partial_x [u^n \partial_x u]$ term appears with the opposite sign (see [8]). There, the energy-minimizing steady state is dynamically stable and attracts all solutions of finite entropy. It is strictly positive and exponentially attractive so long as $M(1 + \alpha^2) > 2\pi$. Otherwise, the rate of convergence is limited by a power-law, at least when $n > \frac{3}{2}$. For $M(1 + \alpha^2) = 2\pi$, the minimizer has a touchdown zero, and for $M(1 + \alpha^2) < 2\pi$, it has the shape of a droplet with zero contact angles.

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2 Energy minimizers

In this section, we study the energy landscape over the space of nonnegative functions of a given mass,

$$\mathcal{C}_M = \left\{ u \in L^2(\Omega) \mid u \geq 0, \int_{\Omega} u(x) dx = M \right\}.$$

We use two topologies on this space, the usual L^2 -distance $\|u - v\|_2$, and the H^1 -topology, with distance function

$$d_{H^1}(u, v) = \|u_x - v_x\|_2, \quad u, v \in \mathcal{C}_M.$$

Since u and v have the same mass, their difference has mean zero, and this distance is equivalent to the usual H^1 -distance. It is clear from Eq. (1.6) that the energy is continuous in H^1 . By convention, $E(u) = \infty$ if $u \notin H^1$.

For $\alpha < 1$, E is a positive definite quadratic form, and hence strictly convex. This can be seen either from the Wirtinger inequality (see, for example, [15, p. 61]), or by writing the energy in terms of the Fourier series of u as

$$E(u) = \pi \sum_{p \in \mathbb{Z} \setminus 0} (p^2 - \alpha^2) |\hat{u}(p)|^2 - \alpha^2 \frac{M^2}{2\pi} - \pi(\hat{u}(1) + \hat{u}(-1)).$$

For $\alpha = 1$, the energy is convex, but not strictly convex on \mathcal{C}_M , because its Fourier expansion depends linearly on $\hat{u}(\pm 1)$. For $\alpha > 1$, convexity is lost.

We will show that the energy has a unique minimizer on \mathcal{C}_M , and describe its profile. Our first lemma shows that a minimizer exists.

Lemma 2.1 (Existence of minimizers.) *For every $M < \infty$, E attains its minimum on \mathcal{C}_M .*

Proof. Let $u \in \mathcal{C}_M$. Since u is nonnegative and has mean $\frac{M}{2\pi}$, it satisfies

$$\int_{\Omega} u^2 dx \leq M \|u\|_{L^\infty}, \quad \|u\|_{L^\infty} \leq \frac{M}{2\pi} + \sqrt{\pi} \|u_x\|_{L^2}.$$

Inserting these estimates into the functional, we obtain

$$E(u) \geq \frac{1}{2} \left(\|u_x\|_{L^2} - \frac{\alpha^2 M \sqrt{\pi}}{2} \right)^2 - \frac{\alpha^4 \pi}{8} M^2 - \left(1 + \frac{\alpha^2}{4\pi} \right) M,$$

which shows that E is bounded below on \mathcal{C}_M .

Consider a minimizing sequence $\{u_j\}_{j \geq 1}$. By Eq. (2.1), the sequence is bounded in H^1 . We invoke the Rellich lemma and pass to a subsequence (again denoted by $\{u_j\}$) that converges weakly in H^1 and strongly in L^2 to some function u^* in \mathcal{C}_M . Since E is weakly lower semicontinuous on H^1 , we have

$$\inf_{u \in \mathcal{C}_M} E(u) \leq E(u^*) \leq \lim_{j \rightarrow \infty} E(u_j) \leq \inf_{u \in \mathcal{C}_M} E(u),$$

and conclude that E attains its minimum at u^* . \square

The Euler-Lagrange equation for E under the mass constraint is given by

$$u_{xx} + \alpha^2 u + \cos x = \lambda, \tag{2.1}$$

where λ is a Lagrange multiplier. We need to incorporate also the positivity constraint. If $u \in \mathcal{C}_M$, we decompose Ω according to the value of u into the positivity set and the zero set of u , defined by

$$P(u) = \{x \in \Omega \mid u(x) > 0\}, \quad Z(u) = \{x \in \Omega \mid u(x) = 0\}.$$

Lemma 2.2 (Euler-Lagrange equation and zero contact angle.) *If u^* minimizes E on \mathcal{C}_M , then it solves (2.1) on $P(u^*)$. The Lagrange multiplier is positive and satisfies $\lambda \geq \sup\{\cos x \mid x \in Z(u^*)\}$. Furthermore, u^* is of class $\mathcal{C}^{1,1}$, and $u_x^* = 0$ on $\partial P(u^*)$.*

Proof. Let ϕ be a smooth 2π -periodic test function, and set

$$u^\varepsilon = \frac{M}{M^\varepsilon} (u^* + \varepsilon \phi), \quad M^\varepsilon = \int_{\Omega} (u^* + \varepsilon \phi) dx.$$

We compute the first variation of E about u^* as

$$\frac{d}{d\varepsilon} E(u^\varepsilon) \Big|_{\varepsilon=0} = \int_{\Omega} (u_x^* \phi_x - \alpha^2 u^* \phi - \phi \cos x + \lambda \phi) dx,$$

where

$$\lambda = -\frac{1}{M} \left(2E(u^*) + \int_{\Omega} u^* \cos x dx \right).$$

Let ϕ be a smooth test function supported in $P(u)$, and set

$$\varepsilon^0 = \left(\max_{x \in \text{supp} \phi} \frac{|\phi(x)|}{u^*(x)} \right)^{-1} > 0.$$

By construction, $u^\varepsilon(x) \geq 0$ for $|\varepsilon| \leq \varepsilon_0$, and therefore the first variation of E along u^ε must vanish. Since this holds for every smooth test function supported on $P(u^*)$, the Euler-Lagrange equation holds there.

Similarly, the first variation is nonnegative for every nonnegative test function ϕ , because positive values of ε yield admissible competitors u^ε on \mathcal{C}_M . This means that u^* satisfies the variational inequality

$$u_{xx} + \alpha^2 u + \cos x \leq \lambda \quad \text{on } \Omega$$

in the sense of distributions. Taking $\phi \equiv 1$, we that $\lambda \geq \alpha^2 \frac{M}{2\pi} > 0$, and by considering nonnegative test functions supported on $Z(u^*)$, we obtain the claimed inequality for λ .

To see that the contact angles are zero, assume that $u^*(\tau) = 0$. The variational inequality implies that $u^*(x) - \frac{\lambda+1}{2}(x-\tau)^2$ is concave in x . In particular, the graph of u^* lies below a support line at $x = \tau$. Solving for $u^*(x)$, we see that

$$0 \leq u^*(x) \leq \frac{\lambda+1}{2}(x-\tau)^2 + b(x-\tau),$$

where b is the slope of the support line. It follows that $b = 0$, and u^* is differentiable at τ with $u_x^*(\tau) = 0$. \square

Our next goal is to determine the minimizers of E on \mathcal{C}_M explicitly. At first sight, this appears to be a simple matter of minimizing over the two free parameters in the general solution of the Euler-Lagrange equation. This solution is given by

$$u(x) = \frac{\lambda}{\alpha^2} + u^0(x) + A \cos(\alpha x) + B \sin(\alpha x), \quad (2.2)$$

where

$$u^0(x) = \begin{cases} -\frac{1}{2}x \sin x, & \alpha = 1, \\ \frac{1}{1-\alpha^2} \cos x, & \alpha \neq 1. \end{cases} \quad (2.3)$$

A moment's consideration shows that the minimizer cannot be a strictly positive function given by Eq. (2.2), unless $A = B = 0$ and $M > \frac{2\pi}{|1-\alpha^2|}$ (in which case $\lambda = \frac{M\alpha^2}{2\pi}$). If u vanishes somewhere in Ω , then λ depends implicitly on M through a nonlinear equation. The number of components of the positivity set is another unknown, the constants A and B may differ from component to component, and the contact points that form the boundary of the positivity set contribute additional free parameters.

We reduce the number of parameters by observing that minimizers of E on \mathcal{C}_M are necessarily symmetric decreasing on $[-\pi, \pi]$ about $x = 0$. To see this, let $u^\#$ be the symmetric decreasing rearrangement of u [26, Section 3.3]. By definition, each sub-level set $\{x \in \Omega \mid u^\#(x) > s\}$ is an open interval centered at $x = 0$ that has the same measure as the corresponding set $\{x \in \Omega \mid u(x) > s\}$. Classical results ensure that $u^\# \in \mathcal{C}_M$, and that

$$\|u^\#\|_{L^2} = \|u\|_{L^2}, \quad \|u_x^\#\|_{L^2} \leq \|u_x\|_{L^2}, \quad \int_\Omega u^\# \cos x \, dx \geq \int_\Omega u \cos x \, dx, \quad (2.4)$$

see [22, 32]. It follows that

$$E(u^\#) \leq E(u).$$

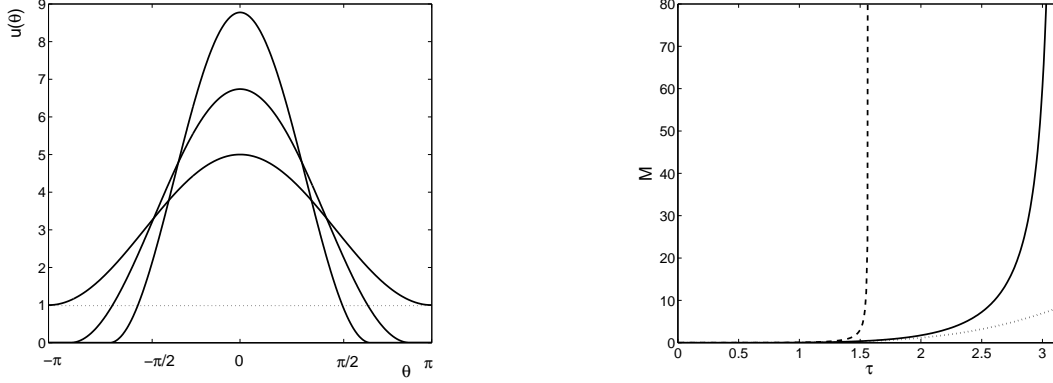


Figure 3: The energy minimizer for $\alpha = 0.5, 1, 2$ with initial data $u_0 = 3$ (left). Mass versus contact point for $\alpha = 2, 1, 0.5$ (right).

If u is a minimizer, then $E(u^\#) = E(u)$, and in particular, the last inequality in (2.4) must hold with equality. Since the cosine is strictly symmetric decreasing, this forces u to be symmetric decreasing as well [26, Theorem 3.4].

It is now easy to determine the minimizer by solving the Euler-Lagrange equation with zero contact angle boundary conditions on a symmetric interval $(-\tau, \tau)$. The remaining three parameters are the coefficient A , the Lagrange multiplier λ , and the contact point τ . The next lemma will be used to show that the positivity set of a minimizer grows with its mass.

Lemma 2.3 (One-sided derivatives.) *Assume that u^* minimizes the energy on \mathcal{C}_M . If $P(u^*) = (-\tau, \tau)$ for some $\tau \in (0, \pi]$, then $u_{xx}^*(\tau_-) > 0$ and $\lambda > \cos \tau$.*

Proof. Since u^* satisfies the Euler-Lagrange equation by Lemma 2.2, it has bounded derivatives of all orders on $(-\tau, \tau)$. We analyze the sign of the first non-vanishing one-sided derivative of u^* . By symmetry, it suffices to consider the right endpoint at $x = \tau$. The positivity of u^* implies that $u_{xx}^*(\tau_-) \geq 0$.

Suppose that $u_{xx}^*(\tau_-) = 0$, then $u_{xxx}^*(\tau_-) \leq 0$. On the other hand, by differentiating Eq. (2.1), we obtain $u_{xxx}^*(\tau_-) = \sin \tau \geq 0$. This leaves only the possibility that $\tau = \pi$. Differentiating once more, we obtain $u_{xxxx}^*(\pi_-) = -1$, which is the wrong sign for u^* to have a minimum at π . It follows that $u_{xx}^*(\tau_-) > 0$. The claimed inequality for λ follows from Eq. (2.1). \square

The following theorem summarizes our results, see Figure 3.

Theorem 2.4 (Description of the energy minimizers.) *Let E be the energy functional in Eq. (1.6) for some $\alpha > 0$. For each $M > 0$, E has a unique nonnegative minimizer of mass M . The minimizer is strictly symmetric decreasing on its positivity set. It depends continuously on M in $\mathcal{C}^{1,1}$, and increases pointwise with M in the sense that for any pair of minimizers u_1^*, u_2^* of mass M_1, M_2 ,*

$$M_1 < M_2 \implies u_1^*(x) < u_2^*(x), \quad x \in P(u_1^*).$$

If $\alpha < 1$ and $M(1 - \alpha^2) > 2\pi$, then the minimizer is strictly positive and given by

$$u^*(x) = \frac{M}{2\pi} + \frac{1}{1 - \alpha^2} \cos x, \quad x \in \Omega. \quad (2.5)$$

Otherwise, it is given by

$$u^*(x) = u^0(x) + A \cos(\alpha x) - u^0(\tau) - A \cos(\alpha \tau), \quad |x| < \tau \quad (2.6)$$

and vanishes elsewhere. Here, τ is a smooth increasing function of M with $\tau \cdot \max\{\alpha, 1\} < \pi$, the function u^0 is given by Eq. (2.3), and $A = \frac{u_x^0(\tau)}{\alpha \sin(\alpha \tau)}$.

Proof. Fix $\alpha > 0$ and $M > 0$. By Lemma 2.1, there exists a minimizer u^* of mass M . If u^* is strictly positive, then Eq. (2.2) holds for all $x \in \Omega$. Since positive minimizers are symmetric about $x = 0$, smooth, periodic, and have mass M , we conclude that $\alpha < 1$ and Eq. (2.5) holds. In order for u^* to be nonnegative and symmetric decreasing we must have $M(1 - \alpha^2) \geq 2\pi$. In that region, u^* is clearly strictly increasing in M .

If, on the other hand, the positivity constraint is active, then u^* is symmetric decreasing on some interval $(-\tau, \tau)$ and vanishes for $|x| \geq \tau$. By Lemma 2.2, $u^* \in C^{1,1}(\Omega)$ and $u_x^*(\pm\tau) = 0$. On $(-\tau, \tau)$, u^* is given by Eq. (2.2). Since u^* and u^0 are even, $B = 0$. The Dirichlet condition at τ yields

$$\lambda = -\alpha^2(A \cos(\alpha \tau) + u^0(\tau)),$$

the Neumann condition determines A , and we find that Eq. (2.6) holds. We denote this function by $u^*(x; \tau)$. If $\tau \cdot \max\{\alpha, 1\} < \pi$, we claim that $u^*(x; \tau)$ is indeed nonnegative, symmetric decreasing in x , and strictly increasing with τ . To see this, we differentiate Eq. (2.6), and use that $u_x^*(\tau; \tau) = 0$ to obtain

$$\frac{dA}{d\tau} \cdot \alpha \sin \alpha \tau = -u_{x\tau}^*(\tau; \tau) = u_{xx}^*(\tau; \tau) > 0.$$

By the chain rule, and using once more that $u_x^*(\tau; \tau) = 0$, we have

$$u_\tau^*(x; \tau) = \frac{dA}{d\tau} \cdot (\cos(\alpha x) - \cos(\alpha \tau)) > 0 \quad \text{for } |x| < \tau.$$

Since u^* vanishes identically when $\tau = 0$, this confirms that it is positive and strictly symmetric decreasing for $|x| < \tau$.

The mass of u^* is given by $M(\tau) = \int_{-\tau}^{\tau} u^*(x; \tau) dx$. We use that $u_x^*(\tau; \tau) = 0$ to compute

$$\frac{dM}{d\tau} = \frac{dA}{d\tau} \int_{-\tau}^{\tau} (\cos(\alpha x) - \cos(\alpha \tau)) dx > 0,$$

and infer that we can solve for $\tau = \tau(M)$ as a strictly increasing smooth function of M . By the chain rule and the inverse function theorem,

$$\frac{d}{dM} u^*(x; \tau(M)) = \frac{\cos(\alpha x) - \cos(\alpha \tau)}{\int_{-\tau}^{\tau} (\cos(\alpha x') - \cos(\alpha \tau)) dx'} > 0.$$

It remains to determine the ranges where Eq. (2.5) and (2.6) hold. For $\alpha < 1$, the energy minimizer on \mathcal{C}_M is unique by the strict convexity of E . If $M \geq \frac{2\pi}{1-\alpha^2}$, the minimizer is given by Eq. (2.5). For smaller values of the mass, we use instead Eq. (2.6), and compute that $M \rightarrow 0$ as $\tau \rightarrow 0$ and $M \rightarrow \frac{2\pi}{1-\alpha^2}$ as $\tau \rightarrow \pi_-$. Continuous dependence on M follows, since Eq. (2.6) agrees with Eq. (2.5) at $M = \frac{2\pi}{1-\alpha^2}$.

When $\alpha \geq 1$, the positivity constraint is always active, because E is not bounded below without it. Therefore, the minimizer is given by Eq. (2.6) on some interval $(-\tau, \tau)$. For $\alpha = 1$, we must have $\tau < \pi$, because the particular solution $u^0(x) = -\frac{1}{2}x \sin x$ from Eq. (2.3) cannot be continued as a differentiable 2π -periodic function across $x = \pi$, in violation of Lemma 2.2. It is easy to check from Eq. (2.6) that $M \rightarrow 0$ as $\tau \rightarrow 0$ and $M \rightarrow \infty$ as $\tau \rightarrow \pi$. For $\alpha > 1$, we have that necessarily $\alpha\tau < 1$, since otherwise the function defined by Eq. (2.6) fails to be symmetric decreasing. Since $M \rightarrow 0$ as $\tau \rightarrow 0$ and $M \rightarrow \infty$ as $\tau \rightarrow \alpha^{-1}\pi$, the theorem follows. \square

3 Steady states

In this section, we investigate the relationship between steady states of Eq. (1.4) and critical points of the energy in Eq. (1.6). For $\alpha \leq 1$ and $n \geq 1$, we will show that the global minimizer u^* determined in Theorem 2.4 is the unique point in \mathcal{C}_M where the energy dissipation defined in Eq. (1.7) vanishes, but for $\alpha > 1$ there are additional steady states. We start with some definitions.

Definition 3.1 Let $u \in \mathcal{C}_M$ such that $E(u) < \infty$, and let $P(u)$ be its positivity set.

- $u \in H^2(\Omega)$ is a *strong steady state* of Eq. (1.4) if for every smooth 2π -periodic test function ϕ

$$\int_{\Omega} (u_{xx} + \alpha^2 u + \cos x) \cdot (u^n \phi_x)_x dx = 0;$$

- u is an L^2 -critical point of the energy on \mathcal{C}_M if every differentiable curve $\gamma : (-\varepsilon_0, \varepsilon_0) \rightarrow \mathcal{C}_M$ in L^2 with $\gamma(0) = u$ satisfies

$$E(\gamma(\varepsilon)) \geq E(u) - o(\|\gamma(\varepsilon) - u\|_{L^2}), \quad \text{as } \varepsilon \rightarrow 0.$$

Strong steady states are time-independent strong solutions in the sense of Eq. (1.5). As elements of H^2 , they are of class $\mathcal{C}^{1,1}$ and meet dry regions with zero contact angles. The reason why we define critical points in the L^2 -topology rather than in H^1 is that the boundary of \mathcal{C}_M in H^1 , which consist of nonnegative functions of mass M that vanish at some point on Ω , contains many curves that are differentiable in L^2 but not in H^1 . By analogy with the subdifferential in convex analysis, we ask only for a lower bound on the energy difference because E is lower semicontinuous, but not continuous, on \mathcal{C}_M with the L^2 -norm.

The following two lemmas relate these notions to the Euler-Lagrange equation.

Lemma 3.2 (Steady states with zero dissipation.) Let $u \in \mathcal{C}_M$, and define the dissipation $D(u)$ by Eq. (1.7). If either

- u satisfies Eq. (2.1) on each component C of $P(u)$ with a constant $\lambda = \lambda(C)$ and with $u = u_x = 0$ on ∂C ,

or, equivalently,

- $u \in H_{loc}^3(P(u)) \cap H^2(\Omega)$ with $u = u_x = 0$ on $\partial P(u)$ and $D(u) = 0$,

then u is a strong steady state.

Proof. We first prove the equivalence of the two conditions. Assume that $u \in \mathcal{C}_M$ satisfies Eq. (2.1) on each component of $P(u)$. Since $u_{xxx} = -\alpha^2 u_x + \sin x \in H^1(P(u))$, it follows that $u \in H_{loc}^3(P(u)) \cap H^2(\Omega)$, and D vanishes. Conversely, if $D(u) = 0$ then $u_{xxx} + \alpha^2 u_x - \sin x$ vanishes in $L_{loc}^2(P(u))$. This means that $u_{xx} + \alpha^2 u + \cos x$ is locally constant on $P(u)$, i.e., u satisfies Eq. (2.1) on each component C of $P(u)$ with a constant $\lambda = \lambda(C)$.

Let $\{C_j\}$ be the collection of connected components of $P(u)$. If u solves Eq. (2.1) on each C_j with some constant λ_j and ϕ is a smooth 2π -periodic test function, then

$$\left| \int_{P(u)} (u_{xx} + \alpha^2 u + \cos x) \cdot (u^n \phi_x)_x dx \right| \leq \sum_j \left| \int_{C_j} \lambda_j \cdot (u^n \phi_x)_x dx \right| = 0,$$

showing that u is a strong steady state. \square

Although strong solutions need not be regular enough to justify differentiating the energy, for $n \geq 1$ it is not hard to show (by arguments analogous to [17, Lemmas 1 and 2]) that the dissipation vanishes in all strong steady states of Eq. (1.7). The next lemma generalizes the description of the minimizers in Lemma 2.2.

Lemma 3.3 (Characterization of critical points.) *A function $u \in \mathcal{C}_M$ is an L^2 -critical point of the energy if and only if there exists a $\lambda \in \mathbb{R}$ such that u solves the Euler-Lagrange equation (2.1) with this value of λ on every component of $P(u)$ and $u = u_x = 0$ on $\partial P(u)$.*

Proof. Suppose that $u \in \mathcal{C}_M$ solves Eq. (2.1) on $P(u)$, and that $u = u_x = 0$ on $\partial P(u)$. For $v \in \mathcal{C}_M$, we compute the directional derivative

$$\begin{aligned} \frac{d}{ds} E((1-s)u + sv) \Big|_{s=0^+} &= \int_{\Omega} \{u_x \cdot (v-u)_x - (\alpha^2 u + \cos x) \cdot (v-u)\} dx \\ &= - \int_{P(u)} \lambda \cdot (v-u) dx - \int_{Z(u)} \cos x \cdot (v-u) dx \\ &= \int_{Z(u)} v \cdot (\lambda - \cos x) dx. \end{aligned}$$

Since E agrees with its second order Taylor expansion about u , it can be written as

$$E(v) = E(u) + \int_{Z(u)} v \cdot (\lambda - \cos x) dx + \frac{1}{2} \int_{\Omega} \{(v_x - u_x)^2 - \alpha^2 (v-u)^2\} dx. \quad (3.1)$$

Let $\gamma : (-\varepsilon_0, \varepsilon_0) \rightarrow \mathcal{C}_M$ be a differentiable curve through u in L^2 . Writing $\gamma(\varepsilon) = u + \varepsilon \gamma'(\varepsilon) + o(\varepsilon)$ in L^2 , we see that the nonnegativity of $\gamma(\varepsilon)$ implies that $\gamma'(\varepsilon)$ vanishes almost everywhere on the zero set $Z(u)$. By Eq. (3.1),

$$E(\gamma(\varepsilon)) - E(u) \geq \int_{Z(u)} \gamma(\varepsilon) \cdot (\lambda - \cos x) dx - \frac{1}{2} \int_{\Omega} \alpha^2 (\gamma(\varepsilon) - u)^2 dx = o(\varepsilon),$$

showing that u is an L^2 -critical point.

Conversely, assume that u is an L^2 -critical point. By considering $\gamma(\varepsilon) = \frac{M}{M\varepsilon}(u + \varepsilon\phi)$, where ϕ is a smooth function with support in $P(u)$, we see that u satisfies the Euler-Lagrange equation on

$P(u)$. Since $u \in H^1$, it is continuous and vanishes on $\partial P(u)$. We need to show that u_x also vanishes on $\partial P(u)$.

Consider a connected component C of $P(u)$, and let ℓ be its length. By Rolle's theorem, u_x vanishes somewhere on C , and by the Euler-Lagrange equation, $\sup_C |u_x| \leq \ell \cdot (\lambda + 1 + \|u\|_\infty)$. In particular, u is Lipschitz continuous on Ω . We claim that u has one-sided derivatives at every point τ with $u(\tau) = 0$. If τ is the limit of an increasing sequence of zeroes of u , then it follows from the above estimate that $u_x(\tau_-) = 0$. Otherwise, if τ lies on the right boundary of a component C , its left derivative exists because u solves the Euler-Lagrange equation on C . Similarly, $u_x(\tau_+)$ exists, and vanishes unless τ is the left endpoint of a component of $P(u)$.

Let ϕ be a smooth test function on Ω , and consider variations of the form

$$u^\varepsilon(x) = \frac{M}{M^\varepsilon} u(x + \varepsilon \phi(x)), \quad M^\varepsilon = \int_\Omega u(x + \varepsilon \phi(x)) dx.$$

Since $u \in H^1$, the curve $\gamma : u \mapsto u^\varepsilon$ is differentiable in L^2 with $\gamma'(0) = \phi u_x$. We compute with the chain rule

$$\left. \frac{d}{d\varepsilon} E(u^\varepsilon) \right|_{\varepsilon=0} = \int_\Omega \phi_x \cdot \left(\frac{1}{2} u_x^2 + \frac{1}{2} \alpha^2 u^2 + u \cos x \right) dx - \int_\Omega \phi u \sin x dx - \lambda \left. \frac{d}{d\varepsilon} M^\varepsilon \right|_{\varepsilon=0},$$

where

$$\lambda = -\frac{1}{M} \left(2E(u) + \int_\Omega u \cos x dx \right), \quad \left. \frac{d}{d\varepsilon} M^\varepsilon \right|_{\varepsilon=0} = \int_\Omega \phi u_x.$$

Setting the first variation equal to zero yields the (weak) Beltrami identity associated with Eq. (2.1) (see, for example [15, Theorem 2.8]). We next write $P(u)$ as the union of its connected components C_j and integrate the first integral by parts on each component. (The number of components may be finite or countable.) Using that u satisfies Eq. (2.1) and vanishes on $\partial P(u)$, we obtain

$$\left. \frac{d}{d\varepsilon} E(u^\varepsilon) \right|_{\varepsilon=0} = \frac{1}{2} \sum_j \phi u_x^2 \Big|_{\partial C_j}.$$

Since u is an L^2 -critical point, this expression vanishes for every smooth function ϕ on Ω . By concentrating ϕ at one point $\tau \in \partial P(u)$, we conclude that $|u_x(\tau_-)| = |u_x(\tau_+)|$.

It remains to show that $u_x(\tau_+) = 0$. Suppose for the contrary that $u_x(\tau_+) = -u_x(\tau_-) = a > 0$. Then τ is the common boundary point of two components of $P(u)$. Let I be an open interval that contains τ but no other zeroes of u , and consider the variation

$$u^\varepsilon(x) = \begin{cases} \frac{M}{M^\varepsilon} \psi^\varepsilon(u(x)), & x \in I, \\ \frac{M}{M^\varepsilon} u(x), & \text{otherwise,} \end{cases} \quad M^\varepsilon = \int_{I^c} u(x) dx + \int_I \psi^\varepsilon(u(x)) dx,$$

where $\varepsilon < \min_{x \in \partial I} u(x)$, and ψ^ε is defined for $y \in [0, \infty)$ by

$$\psi^\varepsilon(y) = \begin{cases} y, & y \geq |\varepsilon|, \\ \varepsilon, & \varepsilon > 0, y \leq \varepsilon, \\ \max\{2y + \varepsilon, 0\}, & \varepsilon < 0, y \leq -\varepsilon. \end{cases}$$

Then $\gamma : \varepsilon \mapsto u^\varepsilon$ defines a curve in \mathcal{C}_M that is differentiable in L^2 with $\gamma(0) = u$, and we have

$$\gamma'(0) = 0, \quad \left. \frac{d}{d\varepsilon} E(\gamma(\varepsilon)) \right|_{\varepsilon=0} = \frac{a}{2},$$

contradicting the assumption that u is an L^2 -critical point. \square

Combining Lemma 3.2 with Lemma 3.3, we see that the profile of a steady state where the dissipation vanishes is the sum of profiles of critical points of the energy whose supports are mutually disjoint. We next show that all these profiles are symmetric.

Lemma 3.4 (Symmetry.) *If u is a positive solution of Eq. (2.1) on some interval C with boundary values $u = u_x = 0$, then it is symmetric about $x = 0$.*

Proof. Let u be a positive solution of Eq. (2.1) on $C = (\tau_1, \tau_2)$. Then u is given by Eq. (2.2). For $\alpha = 1$, the boundary conditions $u(\tau_j) = u_x(\tau_j) = 0$ read

$$\begin{aligned} A \cos(\tau_j) + B \sin(\tau_j) + \lambda - \frac{1}{2}x \sin(\tau_j) &= 0, \\ -A \sin(\tau_j) + B \cos(\tau_j) - \frac{1}{2} \sin(\tau_j) - \frac{1}{2}x \cos(\tau_j) &= 0. \end{aligned}$$

After eliminating B , we see that $\cos \tau_j$ for $j = 1, 2$ both solve the quadratic equation

$$x^2 - 2\lambda x - c = 0, \quad (3.2)$$

where the constant c is determined by A and λ . For $\alpha \neq 1$, the boundary conditions can be expressed in the single complex equation

$$e^{-i\alpha\tau_j}(A + iB) = -\frac{\lambda}{\alpha^2} - \frac{1}{1 - \alpha^2} \cos(\tau_j) + \frac{i}{\alpha(1 - \alpha^2)} \sin(\tau_j).$$

By considering the square modulus $A^2 + B^2$, we see again that $\cos \tau_j$ for $j = 1, 2$ both solve the quadratic equation Eq. (3.2), with a constant that depends on A, B, α , and λ .

By Vieta's theorem, the two roots of Eq. (3.2) add up to 2λ . Since $\lambda \geq \cos \tau_j$ by Lemma 2.2, we conclude that $\cos \tau_1 = \cos \tau_2$. This leaves two scenarios: The first is that $C = (-\tau, \tau)$ for some $\tau \in (0, \pi)$, and u is given by Eq. (2.6). The other scenario is that $C = \Omega \setminus [-\tau, \tau]$ for some $\tau \in [0, \pi)$, and u is the 2π -periodic continuation of

$$u(x) = u^0(x) + A \cos(\alpha(x + \pi)) - u^0(\tau) - A \cos(\alpha(\tau + \pi)), \quad \tau < x < 2\pi - \tau, \quad (3.3)$$

with coefficient $A(\tau) = \frac{u_x^0(\tau)}{\alpha \sin(\alpha(\pi + \tau))}$. In both cases, u is symmetric about $x = 0$. \square

We conclude from Lemmas 3.2-3.4 that Eq. (1.4) has four types of strong steady states with zero dissipation, see Figure 4. The first three are L^2 -critical points of the energy, but the last generally is not. While critical points are typically isolated, the fourth type of steady states can form a one-parameter continuum in \mathcal{C}_M .

- I. *Smooth films* are positive except possibly for touchdown zeroes. They occur as energy minimizers if $\alpha < 1$ with $M(1 - \alpha^2) \geq 2\pi$, and as saddle points if $\alpha > 1$ with $M(\alpha^2 - 1) \geq 2\pi$. If α is not an integer, they are symmetric about $x = 0$ and given by Eq. (2.5). If α is an integer $k > 1$ and $M(k^2 - 1) > 2\pi$, there are also non-symmetric solutions of the form

$$u(x) = \frac{M}{2\pi} - \frac{1}{k^2 - 1} \cos x + A \cos kx + B \sin kx$$

with A and B small enough.

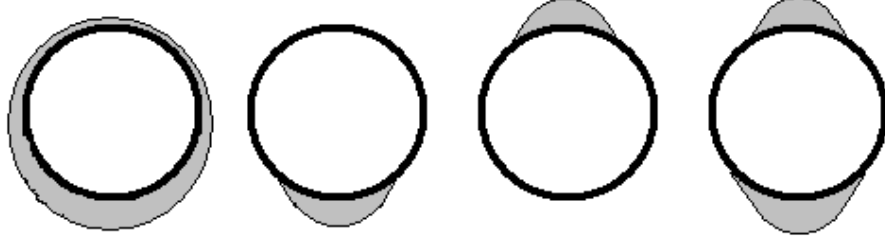


Figure 4: Types of steady states with zero dissipation (*from left to right*): Smooth film, hanging drop, sitting drop, and two-droplet steady state. If $\alpha = 1$, the unique steady state is the energy minimizer, which has the shape of a hanging drop. If $\alpha < 1$, the unique energy-minimizing steady state can take the shape of a hanging drop (for small values of the mass) or a smooth film (for larger values of the mass). If $\alpha > 1$ the energy minimizer is always a hanging drop. For larger values of the mass, there are multiple steady states that include sitting drops and two-droplet steady states.

- II. *Hanging drops* are given by Eq. (2.6) on an interval $(-\tau, \tau)$ with $0 < \tau < \pi$, and have a dry region at the top of the cylinder. Among them are the energy minimizers for $\alpha < 1$ with $M(1 - \alpha^2) < 2\pi$, as well as for $\alpha \geq 1$.

The other two types occur only for $\alpha > 1$:

- III. *Sitting drops* are given by Eq. (3.3) on an interval $(\tau, 2\pi - \tau)$ with $0 < \tau < \pi$, and have a dry region at the bottom. They are L^2 -critical points, but never minimizers of the energy.
- IV. *Two-droplet steady states* are the sum of a hanging and a sitting drop whose positivity sets are disjoint. They do not correspond to L^2 -critical points of the energy unless the value of λ from Eq. (2.1) agrees in the two droplets.

The next theorem is illustrated in Figure 5.

Theorem 3.5 (Uniqueness of steady states with zero dissipation.) *Let $M > 0$ and $\alpha > 0$ be given. If $\alpha \leq 1$, then the global minimizer u^* is the unique strong steady state of Eq. (1.4) on \mathcal{C}_M with zero dissipation. For $\alpha > 1$, there exists for each $M > 0$ an energy level $E_1 > E(u^*)$ such that u^* is the unique strong steady state with zero dissipation in the sub-level set $\{u \in \mathcal{C}_M \mid E(u) < E_1\}$.*

Proof. By Lemmas 3.2-3.4, every strong steady state where the dissipation vanishes is the sum of one or two critical points with disjoint positivity sets. For $\alpha \leq 1$, the convexity of the energy implies that the unique minimizer u^* (determined in Theorem 2.4) is the unique critical point. For $\alpha > 1$, we use that the minimizer is unique and given by a hanging drop, and then use Lemma 3.4 to see that it is isolated within the set of steady states with zero dissipation. \square

4 Convergence to equilibrium

In this section we will prove our main result, that global strong solutions of Eq. (1.4) with $\alpha \leq 1$ converge strongly in H^1 to the unique energy minimizer of the same mass. We have already described these steady states in the previous sections.

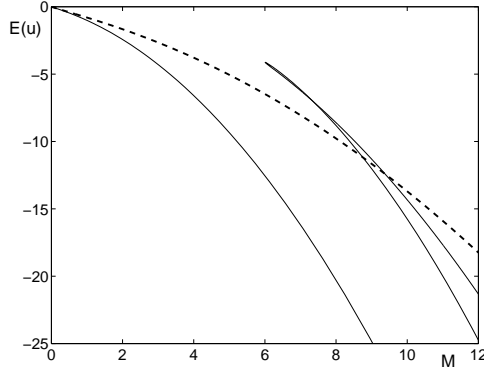


Figure 5: Energy levels of critical points. Dashed line: $\alpha = 1$. For each given mass, the energy minimizer is the unique critical point. Its energy decreases strictly with the mass. Solid line: $\alpha = \sqrt{2}$. If the mass exceeds a certain threshold, there is in addition to the global minimizer a pair of saddle points. Note that the gap between the minimal energy and the energy of the saddles appears to be minimal when the saddles first appear, at $M = 2\pi$. Although the energy levels of the two branches cross near $M = 8$, this is not a bifurcation point.

As described in the introduction, a global strong solution of Eq. (1.4) is a nonnegative function in $L^2_{loc}((0, \infty), H^2(\Omega))$ that satisfies Eq. (1.5) for every smooth test function with compact support in $(0, \infty) \times \Omega$. We consider only strong solutions that additionally satisfy a linear bound on the H^2 -norm,

$$\int_0^T \int_{\Omega} u_{xx}^2 dx dt \leq A + BT \quad (4.1)$$

with some constants A, B , and the *energy inequality*

$$E(u(\cdot, T)) + \int_0^T \int_{P(u)} u^n (u_{xxx} + \alpha^2 u_x - \sin x)^2 dx dt \leq E(u(\cdot, 0)). \quad (4.2)$$

To convince the reader that this class of solutions is not empty, we paraphrase the existence theory of Bernis and Friedman for $n > 2$. We note in passing that the method has been extended to the entire range $n > 0$, and that it implies much stronger existence and regularity results [4, 7, 14].

The basic version of the methods exploits that the entropy

$$S(u) = \int_{\Omega} u^{-n+2} dx$$

formally decreases under Eq. (1.3) with $n > 2$. Note that $S(u) < \infty$ implies that u can vanish only on a set of measure zero. Strictly speaking, S is not an entropy for Eq. (1.4), because it may increase as well as decrease along solutions, due to the presence of the long-wave instability and the gravitational drainage term. These terms can be accommodated with a technique of Chugunova, Pugh and Taranets [14], as follows. Along smooth solutions,

$$c_n^{-1} \frac{dS(u)}{dt} = - \int_{\Omega} u_{xx}^2 dx + \int_{\Omega} (\alpha^2 u_x^2 + u \cos x) dx, \quad (4.3)$$

where $c_n = (n-2)(n-1)$. Integrating over time and bounding the second integral with Eq. (2.1), one obtains

$$S(u(\cdot, T)) + c_n \int_0^T \int_{\Omega} u_{xx}^2 dx dt \leq S(u(\cdot, 0)) + KT, \quad (4.4)$$

where K depends only on the energy and the mass. In particular, classical solutions satisfy Eq. (4.1). A similar computation shows that the energy inequality (4.2) is an identity for classical solutions.

Strong solutions that satisfy Eqs. (4.1) and (4.2) are obtained by regularizing

$$u_t + \partial_x [f_{\epsilon}(u) \partial_x (u_{xx} + \alpha^2 u + \cos x)] = 0, \quad x \in \Omega,$$

for example with $f_{\epsilon}(z) = z^n + \epsilon$ [5]. The regularized equations are known to have unique smooth solutions that exist for all times $t > 0$. For these solutions, the energy inequality (4.2) holds with equality, and the entropy inequality (4.3) holds for the corresponding entropy functional with the same constants c_n and K . By the usual compactness arguments, there exists a sequence u^{ϵ_j} that converges uniformly on compact time intervals to a nonnegative strong solution u on $(0, \infty) \times \Omega$, which is smooth wherever it is positive. Moreover, $u_{xx}^{\epsilon_j}$ converges to u_{xx} weakly in $L^2(0, T) \times \Omega$ for each $T > 0$, and $u_{xxx}^{\epsilon_j}$ converges to u_{xxx} weakly in $L_{loc}^2(P(u))$. In the limit, the energy and entropy inequalities in Eqs. (4.2) and (4.4) remain intact, because the double integrals are weakly lower semicontinuous due to their convexity in the highest derivative.

We turn to the long-time behaviour of solutions. Lyapunov's principle says that a dissipative dynamical system should converge towards the set of critical points of the energy. Thin film equations present two difficulties: Since well-posedness has not been settled, we are not sure how to view them as dynamical systems, and lower bounds on the dissipation are not easy to obtain.

The next lemma bounds the distance from the minimizer in terms of the energy. Since energy decreases along solutions of Eq. (1.4), the lemma implies that u^* is dynamically stable in the sense of Lyapunov.

Lemma 4.1 (local coercivity.) *Let $\alpha > 0$ and $M > 0$ be given, and let u^* be the energy minimizer on \mathcal{C}_M obtained in Theorem 2.4. Then*

$$\sup \left\{ d_{H^1}(u, u^*) \mid u \in \mathcal{C}_M, E(u) \leq E(u^*) + \Delta E \right\} \longrightarrow 0 \quad (\Delta E \rightarrow 0).$$

For $\alpha < 1$, we have the explicit estimate

$$d_{H^1}(u, u^*) \leq \left(\frac{2\Delta E}{1 - \alpha^2} \right)^{1/2} \quad (4.5)$$

for all u with $E(u) \leq E(u^*) + \Delta E$.

Proof. We argue by contradiction. If the conclusion fails, then there exists a minimizing sequence $\{u_j\}$ such that $\inf_j d_{H^1}(u_j, u^*) > 0$. By Lemma 2.1, a subsequence converges weakly in H^1 and strongly in L^2 to the minimizer u^* . But since the minimizer is unique, and all convergent subsequences have the same limit, the entire sequence converges. By the expansion of the energy in Eq. (3.1),

$$\begin{aligned} d_{H^1}(u_j, u^*) &= \left(2\Delta E(u_j) - 2 \int_{Z(u^*)} u_j \cdot (\lambda - \cos x) dx + \alpha^2 \int_{\Omega} (u_j - u^*)^2 dx \right)^{\frac{1}{2}} \\ &\rightarrow 0 \quad (j \rightarrow \infty), \end{aligned}$$

contradicting the choice of the sequence. For $\alpha < 1$, the bound in Eq. (4.5) follows immediately from the observation that the linear term in Eq. (3.1) is nonnegative, see Lemma 2.2. \square

For $\alpha = 1$, one can take advantage of the positivity of the linear term in Eq. (3.1) to obtain an explicit estimate of the form $d_{H^1}(u, u^*) \leq c_1(\Delta E)^{1/2} + c_2\Delta E$, where the constants c_1 and c_2 depend on the mass. We next construct a sequence of times along which the dissipation goes to zero.

Lemma 4.2 (Construction of a weakly convergent sequence.) *Let u be a global strong solution of Eq. (1.4) that satisfies inequalities (4.1) and (4.2), and let E_0 be its energy at time $t = 0$. There exists a sequence of times $t_j \rightarrow \infty$ such that*

$$\sup_j \|u_{xx}(\cdot, t_j)\|_2 < \infty, \quad \lim_{j \rightarrow \infty} D(u(\cdot, t_j)) = 0.$$

Proof. Eq. (4.1) implies that the set

$$C_1 = \left\{ t \in (0, T) \mid \|u_{xx}(\cdot, t)\|_2^2 \geq 4B \right\}$$

has measure bounded by $\mu(C_1) \leq \frac{A}{4B} + \frac{T}{4}$. Similarly, for every $\varepsilon > 0$, Eq. (4.2) implies that

$$C_2 = \left\{ t \in (0, T) \mid D(u(\cdot, t)) \geq \varepsilon \right\}$$

has measure bounded by $\mu(C_2) \leq \frac{E_0 - E(u^*)}{\varepsilon}$. It follows that for $T > \frac{A}{B} + \frac{4(E_0 - E(u^*))}{\varepsilon}$, we can find $t \in [\frac{T}{2}, T]$ that lies neither in C_1 nor in C_2 . The sequence t_j is constructed by taking sequences $\varepsilon_j \rightarrow 0$ and $T_j \rightarrow \infty$. \square

We combine this lemma with the stability result from Lemma 4.1 to show that u^* is in fact asymptotically stable.

Theorem 4.3 (Asymptotic stability.) *Let u be a global strong solution of Eq. (1.4) of mass M and initial energy E_0 constructed by the method of Bernis and Friedman, and let u^* be the energy minimizer on \mathcal{C}_M . If $\alpha > 1$, assume in addition that the sub=level set $\{E \leq E_0\}$ in \mathcal{C}_M contains no other steady states with zero dissipation. Then*

$$\lim_{t \rightarrow \infty} d_{H^1}(u(\cdot, t), u^*) = 0.$$

Proof. Let $\{t_j\}$ be the sequence of times constructed in Lemma 4.2. Since $\{u(\cdot, t_j)\}$ is uniformly bounded in H^2 , there is a subsequence (again denoted t_j) that converges weakly in H^2 and strongly in H^1 to some limit $v \in \mathcal{C}_M$. We want to show that $v = u^*$.

For $\delta > 0$, consider the set $P_\delta(v) = \{x \in \Omega \mid v(x) > \delta\}$. Since $u(\cdot, t_j)$ converges uniformly to v , we have that $u(x, t_j) > \frac{\delta}{2}$ on $P_\delta(v)$ for j sufficiently large, and it follows that

$$\int_{P_\delta(v)} (u_{xxx}(\cdot, t_j) + \alpha^2 u_x(\cdot, t_j) - \sin x)^2 dx \leq \frac{2}{\delta} D(u(\cdot, t_j)) \rightarrow 0.$$

Since we already know that $u_x(\cdot, t_j)$ converges to v_x strongly in L^2 , this means that $u_{xxx}(\cdot, t_j)$ converges to $\sin x - \alpha^2 v_x$ strongly in $L^2(P_\delta(v))$. The limit agrees with v_{xxx} , and we see that

$$v_{xxx} + \alpha^2 v_x - \sin x = 0 \quad \text{on } P_\delta(v).$$

Since δ was arbitrary and $v \in H^2$ by construction, it follows from Lemma 3.2 that v is a strong steady state of Eq. (1.4). Since $E(v) \leq E(u(\cdot, 0))$ and $D(v) = 0$, we conclude with Theorem 3.5 that $v = u^*$.

We next observe that $E(u(\cdot, t_j)) \rightarrow E(u^*)$ by the continuity of the energy in H^1 . Since the energy decreases monotonically along solutions by Eq. (4.2), we have

$$\lim_{t \rightarrow \infty} E(u(\cdot, t)) = E(u^*),$$

and the claimed convergence follows with Lemma 4.1. \square

5 Rate of convergence

In the final section, we consider the rate of convergence to steady states. We will show that strictly positive energy minimizers are exponentially attractive, while steady states that have zeroes can be approached at most at a polynomial rate. The reason is that the entropy inequality in Eq. (4.4) limits the rate at which the solution can converge to zero on a subset of Ω . To obtain the strongest lower bound, we will use Kadanoff's entropy

$$S(u) = \int_{\Omega} u^{-n+\frac{3}{2}} dx. \quad (5.1)$$

One can verify by direct calculation (involving repeated integration by parts [4, 20]) that classical positive solutions of the thin-film equation Eq. (1.3) with $n \neq \frac{3}{2}$ satisfy

$$c_n^{-1} \frac{dS(u)}{dt} = \int_{\Omega} u^{-\frac{1}{2}} u_x u_{xxx} dx = -4 \int_{\Omega} u^{\frac{1}{2}} \left((u^{\frac{1}{2}})_{xx} \right)^2 dx < 0, \quad (5.2)$$

where $c_n = (n - \frac{3}{2})(n - \frac{1}{2})$. This is a special case of Eq. (2.13) in [4]. For Eq. (1.4), Kadanoff's entropy can grow at most linearly with time:

Lemma 5.1 (Entropy bound.) *Fix $n > \frac{3}{2}$, and let S be given by Eq. (5.1). Let $u_0 \in \mathcal{C}_M$ be an initial value of finite energy and entropy. Then the global strong solution of Eq. (1.4) constructed by the method of Bernis and Friedman satisfies*

$$S(u(\cdot, t)) \leq S(u(\cdot, 0)) + K_0 t, \quad (5.3)$$

where K_0 depends on the mass and the initial energy.

Proof. If u is a positive classical solution of Eq. (1.4), we differentiate the entropy and integrate by parts to obtain

$$c_n^{-1} \frac{dS(u)}{dt} = \int_{\Omega} u^{-\frac{1}{2}} u_x u_{xxx} dx + \alpha^2 \int_{\Omega} u^{-\frac{1}{2}} u_x^2 dx - \int_{\Omega} u^{-\frac{1}{2}} u_x \sin x dx,$$

where $c_n = (n - \frac{3}{2})(n - \frac{1}{2})$. The first summand we rewrite with the help of Eq. (5.2). The second summand we integrate by parts

$$\int_{\Omega} u^{-\frac{1}{2}} u_x^2 dx = 2 \int_{\Omega} u (u^{\frac{1}{2}})_{xx} dx,$$

and combine it with the first by completing the square. This produces a remainder term of the form $\frac{\alpha^4}{4} \int_{\Omega} u^{\frac{3}{2}} dx$. The third summand we integrate by parts as well. We arrive at

$$c_n^{-1} \frac{dS(u)}{dt} = - \int_{\Omega} u^{\frac{1}{2}} \left(2(u^{\frac{1}{2}})_{xx} - \frac{\alpha^2}{2} u^{\frac{1}{2}} \right)^2 dx + \frac{\alpha^4}{4} \int_{\Omega} u^{\frac{3}{2}} dx + 2 \int_{\Omega} u^{\frac{1}{2}} \cos x dx.$$

By Lemma 2.1, the last two integrals are bounded by a constant that depends only on the mass and the energy. Integrating along the solution, we see that Eq. (5.3) holds for classical solutions.

By the same computation, Eq. (5.3) holds for the solutions of a suitably regularized equation with the correspondingly regularized entropy and with the same constant K_0 . Since the strong solution is a uniform limit of such solutions, the entropy converges as well, and the claim follows. \square

Theorem 5.2 (Bounds on the rate of convergence.) *Consider Eq. (1.4) with parameters $n > 0$ and $\alpha > 0$, and set $\beta = n - \frac{3}{2}$. Let u be a solution of mass M that satisfies the energy and entropy inequalities in (4.2) and (5.3). Assume that u converges in H^1 to the energy-minimizing steady state u^* of mass M .*

- If $n > \frac{3}{2}$ and u^* vanishes on a set of positive length L , then

$$d_{H^1}(u(\cdot, t), u^*) \geq \frac{1}{\sqrt{\pi}} \cdot \left(\frac{L}{S_0 + K_0 t} \right)^{\frac{1}{\beta}};$$

- if $n > 2$ and u^* vanishes quadratically at a point, then there exist positive constants K_1 and K_2 (depending on the initial energy and entropy) such that

$$d_{H^1}(u(\cdot, t), u^*) \geq (K_1 + K_2 t)^{-\frac{2}{2\beta-1}};$$

- if $\alpha < 1$ and u^* is strictly positive, then

$$d_{H^1}(u(\cdot, t), u^*) \leq K_3 e^{-\mu t}$$

for some constant K_3 (depending on u), where $\mu = (1 - \alpha^2)(\min u^*)^n$.

Proof. If $Z(u^*)$ has measure $L > 0$, we estimate

$$S(u(\cdot, t)) \geq L \left(\sup_{x \in Z(u^*)} u(x, t) \right)^{-\beta} \geq L \cdot \|u(\cdot, t) - u^*\|_{L^\infty}^{-\beta}.$$

Since u and u^* have the same mass, we have $\|u(\cdot, t) - u^*\|_{L^\infty} \leq \sqrt{\pi} d_{H^1}(u(\cdot, t), u^*)$, and the first claim follows from the bound on the entropy in Eq. (5.3).

If u^* has a zero of order $\gamma > \frac{1}{\beta}$, we consider the interval I of length L centered at that point and obtain with the same calculation as for the first case that

$$\|u(\cdot, t) - u^*\|_{L^\infty} \geq \sup_{x \in I} u(x, t) - \sup_{x \in I} u^*(x) \geq \left(\frac{L}{S_0 + K_0 t} \right)^{\frac{1}{\beta}} - O(L^\gamma)$$

as $L \rightarrow 0$. Choosing $L = \varepsilon \cdot (S_0 + K_0 t)^{-\frac{1}{\beta\gamma-1}}$, we see that for $\varepsilon > 0$ sufficiently small

$$\|u(\cdot, t) - u^*\|_{L^\infty} \geq (K_1 + K_2 t)^{-\frac{\gamma}{\beta\gamma-1}}$$

with some constants $K_1, K_2 > 0$. Setting $\gamma = 2$ and adjusting the constants we obtain the second claim.

If u^* is strictly positive, then $\alpha < 1$ by Theorem 2.4. By Eq. (3.1),

$$\begin{aligned} E(u(\cdot, t)) - E(u^*) &= \frac{1}{2} \int_{\Omega} ((u_x - u_x^*)^2 - \alpha^2(u - u^*)^2) dx \\ &= \pi \sum_{p \in \mathbb{Z} \setminus \{0\}} (p^2 - \alpha^2) |\hat{u}(p) - \hat{u}^*(p)|^2. \end{aligned}$$

Since u converges to u^* in H^1 , there exists a time t_0 such that $\min u(\cdot, t) > 0$ for all $t > t_0$. At all later times, u is a strictly positive, classical solution that can be differentiated as often as necessary. Since u^* solves the Euler-Lagrange equation (2.1), the dissipation satisfies

$$\begin{aligned} \frac{d}{dt} E(u(\cdot, t)) &= - \int_{\Omega} u^n ((u - u^*)_{xxx} + \alpha^2(u - u^*)_x)^2 dx \\ &\leq -(\min u)^n \cdot \int_{\Omega} [\partial_x ((u - u^*)_{xx} + \alpha^2(u - u^*))]^2 dx \\ &= -(\min u)^n \cdot \pi \sum_{p \in \mathbb{Z} \setminus \{0\}} p^2 (p^2 - \alpha^2)^2 |\hat{u}(p) - \hat{u}^*(p)|^2 \\ &\leq -\left(\frac{\min u}{\min u^*}\right)^n \cdot 2\mu (E(u(\cdot, t)) - E(u^*)). \end{aligned}$$

In the last two steps, we have used Parseval's identity to rewrite the integral in terms of the Fourier coefficients of $u - u^*$, and estimated the Fourier multipliers by

$$p^2(p^2 - \alpha^2)^2 \geq (1 - \alpha^2)(p^2 - \alpha^2), \quad (p \neq 0).$$

Since $u(\cdot, t)$ converges uniformly to u^* as $t \rightarrow \infty$ by Theorem 4.3, it follows from Gronwall's lemma that $E(u(\cdot, t)) - E(u^*) \leq K e^{-2\mu t}$ for some constant K . By Eq. (4.5) of Lemma 4.1, this implies the claimed exponential convergence of $d_{H^1}(u(\cdot, t), u^*)$. \square

If $n = \frac{3}{2}$ and u^* vanishes on a set of positive length, we obtain as in the proof of the first case of the theorem yields an exponential bound of the form $d_{H^1}(u(\cdot, t), u^*) \leq K e^{-\mu t}$, where K and μ depend on the mass, energy, and entropy of the solution.

Summary (Pukhnachev's model on a stationary cylinder.) *Let u be a global strong solution of*

$$u_t + \partial_x [u^3 \partial_x (u_{xx} + u + \cos x)] = 0, \quad x \in \mathbb{R}/(2\pi\mathbb{Z})$$

constructed by the method of Bernis and Friedman, and let u^ be the unique nonnegative energy minimizer of the same mass M . Then*

$$\lim_{t \rightarrow \infty} d_{H^1}(u(\cdot, t), u^*) = 0.$$

The minimizer is a droplet with zero contact angles and profile

$$u^*(x) = -\frac{1}{2}(x \sin x - \tau \sin \tau) + \frac{1}{2}(1 - \tau \cot \tau)(\cos \tau - \cos x), \quad |x| < \tau.$$

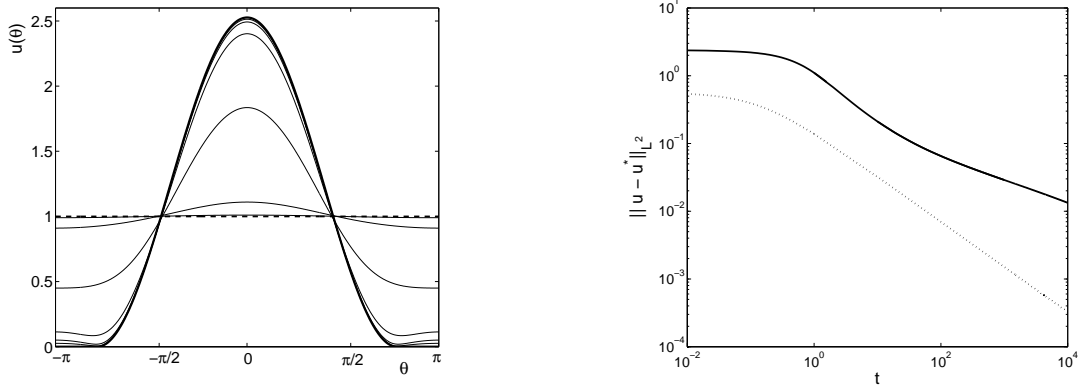


Figure 6: Evolution of a solution with $\alpha = 1$, $n = 3$ and initial data $u_0 = 1$. Time shots of the numerical solution at $t = 0, 10^{-2}, 10^{-1}, 1, 10, 10^2, 10^3$ (left). L^2 -distance of the solution from the energy minimizer. The dashed line shows the lower bound from Eq. (5.4) (right).

The contact point τ is a continuous, strictly increasing function of the mass, with $\tau \rightarrow 0$ as $M \rightarrow 0$ and $\tau \rightarrow \pi$ as $M \rightarrow \infty$. If, additionally,

$$\int_{\Omega} (u(x, t))^{-\frac{3}{2}} dx \leq S_0 + K_0 t,$$

then

$$d_{H^1}(u(\cdot, t), u^*) \geq \frac{1}{\sqrt{\pi}} \cdot \left(\frac{2(\pi - \tau)}{S_0 + K_0 t} \right)^{\frac{2}{3}}. \quad (5.4)$$

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